Reconstruction and Extension of the Family–Vicsek Scaling Hypothesis for Growing Rough Interfaces

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A scaling hypothesis for the standard deviation $\sigma$ of the height of growing interfaces is proposed by extending the Family–Vicsek (FV) scaling hypothesis. A data-collapsing method is adopted for estimating values of three exponents $\alpha$, $\beta$, and $\gamma$, which characterize, respectively, the roughness, growth, and dynamic properties of growing interfaces. The estimation is carried out through the scaling law for a short width, a scaling argument is possible even for growing interfaces that do not satisfy the FV scaling relation. Successful applications are carried out to a few numerical models and a paper-wetting experiment.

KEYWORDS: interface growth, roughness, extended scaling hypothesis, growth exponent, data-collapsing

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1. Introduction

The time evolution of the roughness of interfaces can be widely observed in a variety of growth phenomena, such as crystal growth, viscous flow in porous media, mountain topography, and bacterial colony formation. To characterize rough interfaces, their statistical properties have been focused on. Here, we suppose that an interface exists and moves upward with time, as shown in Fig. 1. We consider a straight line $c(t)$, which is defined by linear approximation of the interface at time $t$. $c(t)$ for all $t$ are assumed to be parallel. Initially at $t = 0$, the interface is assumed to be flat. Then, $c(0)$ is identical to the initial interface. We introduce a line segment with length $L$ on $c(0)$, and the distance from the left end of the segment is denoted by $x$. This line segment $c(x, 0)$ $(0 \leq x \leq L)$ can be used as a window for the observation of subsequent interface growth. In the observation window, we define the height of the interface, denoted by $h$, as the distance between a point on the interface and the line segment $c(x, 0)$. Then, the height of the interface can be described by a function of $x$ and $t$, $h(x, t)$. If the interface has an overhang, $h$ becomes a multiple-valued function with respect to $x$. In such a case, we adopt the maximum value for $h$ so as to make $h(x, t)$ a single-valued function. Using the single-valued function $h(x, t)$, the average $\bar{h}$ and the standard deviation $\sigma$ of $h(x, t)$ are introduced as follows:

$$\bar{h}(L, t) = \frac{1}{L} \int_0^L h(x, t) \, dx,$$

$$\sigma(L, t) = \left( \frac{1}{L} \int_0^L (h(x, t) - \bar{h}(L, t))^2 \, dx \right)^{1/2}.$$  \hspace{1cm} (2)

The roughness of the interface is characterized quantitatively by $\sigma(L, t)$.

In 1985, Family and Vicsek proposed the following scaling hypothesis for the growth of rough interfaces. Let us consider that interface growth starts from a flat interface. In this case, $\sigma$ increases as time proceeds. Family and Vicsek assumed that $\sigma$ increases temporally as $t^\beta$ and finally saturates at a certain value, that depends on the window width $L$ as $L^\gamma$. Then, there exists a crossover time $t^*$ between the growth region and the saturated region. $t^*$ is assumed to be scaled by $L$ as $t^* \sim L^\gamma$. The exponents $\alpha$, $\beta$, and $\gamma$ are called the roughness, growth, and dynamic exponents, respectively. With respect to $\sigma(L, t)$, Family and Vicsek adopted the following scaling relation:

$$\sigma(L, t) \sim L^\alpha f\left(\frac{t}{L^{\beta/\gamma}}\right).$$  \hspace{1cm} (3)

where $f(x)$ is a scaling function, that satisfies

$$f(x) \sim \begin{cases} x^{\beta}, & x \ll 1, \\ 1, & x \gg 1. \end{cases}$$  \hspace{1cm} (4)

When $t \ll t^*$, we obtain $\sigma \sim L^\alpha (t/L^{\beta/\gamma})^{\beta} \sim L^{\alpha-\beta/\gamma}\beta$. Since $\sigma$ is assumed to be independent of $L$ when $t \ll t^*$ in the Family–Vicsek (FV) scheme, the exponents $\alpha$, $\beta$, and $\gamma$ must satisfy the scaling law...
\[ \alpha = \beta z. \]  

In applying the FV scaling hypothesis to some numerical and experimental results, we have found that estimating the value of \( \beta \) is very difficult since it is difficult to obtain \( \sigma(L,t) \) precisely in the early stage of interface growth. Actually, values of \( \beta \) have not been obtained yet in many interface growth experiments. Moreover, there is another difficulty in applying FV scaling: some rough interfaces do not satisfy the FV scaling law. In fact, we cannot find any clear reason why growing interfaces necessarily satisfy the scaling law in eq. (5). Therefore, it is important to extend the FV scaling relation so as to overcome such difficulties.

In the next section, we present a scaling hypothesis, that is an extension of FV scaling. In §3, we report the results obtained when applying the extended scaling hypothesis to some numerical models and a paper-wetting experiment. A summary and conclusion are given in §4.

2. Extension of the Family–Vicsek Scaling Hypothesis

In many previous papers, the values of \( \alpha \) and \( \beta \) were estimated from the slope of \( \sigma(L,t) \) as a function of \( L \) and \( t \) respectively. For the experimental value of \( \beta \), it was found that this estimation caused great difficulty since precise measurement of the time dependence of \( \sigma \) at the initial stage of interface growth \( (t \ll t^\ast) \) was very difficult to carry out. Thus, we propose another formulation of the scaling behavior of \( \sigma(L,t) \). We first consider that \( \sigma(L,t) \) is divided into increasing and saturated regions for \( L \) at the crossover length \( L^\ast \). The relation between \( L^\ast \) and \( t \) is assumed to be

\[ L^\ast \sim t^{1/\gamma}. \]  

As in the case of FV scaling, it is assumed that \( \sigma(L,t) \) grows as \( L^\alpha \) in the increasing region and is independent of \( L \) in the saturated region. We adopt the following scaling relation instead of the original FV scaling,

\[ \sigma(L,t) \sim t^\beta g \left( \frac{L}{t^{1/\gamma}} \right), \]  

where \( g(x) \) is a scaling function satisfying

\[ g(x) \sim \begin{cases} x^\gamma, & x \ll 1, \\ 1, & x \gg 1. \end{cases} \]

On the basis of eqs. (6)–(8), we consider the possible asymptotic behavior of \( \sigma(L,t) \) with respect to \( L \), and the following result is obtained:

\[ \sigma(L,t) \sim \begin{cases} L^\alpha t^\beta, & L \ll L^\ast, \\ t^\beta, & L \gg L^\ast. \end{cases} \]

where \( \gamma = \beta - \alpha/z \).

When \( \gamma = 0 \), \( \alpha = \beta z \) is obtained and the FV scaling law [eq. (5)] is reproduced. Here, however, there is no necessity for \( \gamma \) to be zero in our scaling hypothesis. The exponent \( \gamma \) represents the deviation from FV scaling and indicates the time dependence of \( \sigma \) for small \( L \). Our scaling hypothesis is a natural extension of FV scaling, and \( \gamma = \beta - \alpha/z \) is the extended scaling law. No matter how small the extent, the significance of the reconstruction of the scaling relation lies in the following three advantages: (A) The value of \( \beta \) can be obtained with high accuracy, since we can easily vary the width of the observation window \( L \), and the dependence of \( \sigma \) on \( L \) can be measured precisely. (B) \( \beta \) can be defined not only in the initial stage of interface growth but also in intermediate stages, where the value of \( \beta \) may be different from that in the initial stage. (C) By introducing the exponent \( \gamma \), the extended scaling relation is applicable to some experimental cases where \( \sigma \) does not saturate within a limited observation time. Namely, our method is applicable to the intermediate stages.

Our method for estimating the values of \( \alpha, \beta, \) and \( z \) is as follows.\(^7\) For the data of \( \sigma(L,t) \) as a function of \( L \) for several values of \( t \), we plot \( L(t)^{1/\gamma} \) vs \( \sigma(t)^\beta \) on a double logarithmic chart with varying values of \( \beta \) and \( z \). From the scaling relation given by eq. (7), the values of \( \beta \) and \( z \) are determined so that the plotted curves are collapsed onto a single curve. The single curve obtained by this data-collapsing method corresponds to the scaling function \( g(x) \) in eq. (8), and the value of \( \alpha \) can be estimated from the slope of the single curve when \( L(t)^{1/\gamma} \ll 1 \). In our method, the values of \( \beta \) and \( z \) can be estimated as fitting parameters of the data collapse. Since we do not directly consider the asymptotic behavior of \( \sigma(L,t) \) with respect to \( t \) in the present extended scaling hypothesis, we can advantageous characterize separate scaling behaviors in different time regions during interface growth.

3. Applications

In this section, we report the results of applying the extended scaling hypothesis to some numerical models and a paper-wetting experiment. In particular, examples that illustrate the three advantages (A)–(C) stated in §2, are reported in §3.1–§3.3, respectively.

Regarding the numerical models we adopted, there are two types: \( x \) and \( h(x,t) \) are continuous or discrete. All discrete models were simulated using a square grid. Each site in the grid is labelled \( (j,k) \). The width of the observation window \( L \) and the height of the interface \( h \) are discretized to become integers. The integer \( j \) represents a position in the observation window and satisfies \( 1 \leq j \leq L \). The width of the grid, i.e., the system size, is denoted by \( W \). In each discrete model, a rule for occupation of a particle in an empty (i.e., unoccupied) site in the grid is described. The iteration of the rule corresponds to the time evolution of the domain of occupied sites. The time unit for each discrete model is defined as \( W \) iterations. As time proceeds, the domain grows. The interface is represented by the occupied sites that are adjacent to an empty site. The height of the interface is given as a function of both \( j \) and \( t \), \( h(j,t) \). Assuming that the initial interface is flat and located at \( k = 0 \), the integer \( k \) indicates the distance from the initial interface.

3.1 Confirmation of the FV scaling relation by our method

As examples for advantage (A), we carried out numerical simulation of the models shown in the following subsections. In particular, to show the validity of our method, we compare the values of \( \alpha, \beta, \) and \( z \) obtained by our method with those obtained by the original FV scaling that have been reported previously. The important point is that the value of \( \beta \) can be obtained even if the data for \( \sigma(L,t) \) in terms of \( t \) are few so that its value is not determined precisely from the slope of \( \ln \sigma \) vs \( \ln t \).
3.1.1 Kardar–Parisi–Zhang equation (KPZ eq.)

The KPZ equation

\[ \frac{\partial h(x,t)}{\partial t} = \nu \frac{\partial^2 h(x,t)}{\partial x^2} + \frac{\lambda}{2} \left( \frac{\partial h(x,t)}{\partial x} \right)^2 + \eta(x,t) \quad (10) \]

is a continuum model for describing interface growth.\(^\text{8}\) Here, \(\nu\) represents surface tension and \(\lambda\) indicates the contribution of the tilt of the interface to the time evolution of the height. \(\eta(x,t)\) is a noise term given by the following Gaussian distribution:

\[ \langle \eta(x,t) \rangle = 0, \quad \langle \eta(x,t)\eta(x',t') \rangle = 2D \delta(x-x') \delta(t-t'). \quad (11) \]

In a theoretical study on the scaling behavior of growing interfaces obtained by the KPZ eq., it was reported that \(\alpha = 1/2\) and \(\beta = 1/3\).\(^\text{8}\)

Discretizing eq. (10) with respect to space and time \([x \rightarrow j \Delta x, t \rightarrow n \Delta t, h(x,t) \rightarrow h(j,n)]\), we obtain

\[ h(j,n+1) = h(j,n) + \frac{\Delta t}{\Delta x^2} \left\{ \nu [h(j-1,n) - 2h(j,n) + h(j+1,n)] \right\}
+ \frac{\lambda}{8} [h(j+1,n) - h(j-1,n)]^2 + \sqrt{2D\Delta t} \xi(j,n), \quad (12) \]

where \(j\) and \(n\) are integers. \(\Delta x\) and \(\Delta t\) represent spatial and temporal units, respectively. The parameter values we used were \(\Delta x = 1\), \(\Delta t = 0.02\), \(\nu = 0.5\), \(\lambda = 17.3\), and \(D = 0.005\). Instead of the noise term \(\eta(x,t)\) in eq. (10), we used a uniform random number with the same second moment as \(\eta(x,t)\), which appears in the last term of eq. (12). Here, \(\xi(j,n)\) is a random value taken from a uniform distribution between \(-1\) and \(+1\). The validity of this replacement for the noise was reported in a previous paper.\(^\text{9}\) As the initial condition (at \(n = 0\)), we set \(h = 0\) for all \(j\). The boundary condition for \(j\) is periodic. The system size, i.e., the width of the interface, is denoted by \(W\). Note that the system size \(W\) is different from the window size of the observation \(L\). Hence \(L\) can be freely varied within \(W\), even for one simulation run with a specified value of \(W\).

Figure 2(a) shows a set of snapshots of the fluctuation in \(h\) for different values of \(t\) for \(L = W = 500\), obtained by solving eq. (12). As a preliminary study, we investigated the dependence of \(\sigma\) on \(t\). The result is shown in Fig. 2(b), where we took \(L = W\) for the size of the observation window. To obtain the \(t-\sigma\) plot for each value of \(W\), 100 runs were averaged. From the slope of the \(t-\sigma\) curve in the range of \(t = 10^6\) to \(10^9\) for \(W = 3000\), we obtained \(\beta \approx 0.33\). Figure 3(a) shows \(L-\sigma\) plots for five different values of \(t\). Each curve in this figure was obtained by averaging over 20 runs. Using the data-collapsing method, these \(L-\sigma\) curves can be collapsed onto a single curve, as shown in Fig. 3(b), when \(\beta \approx 0.33\) and \(\gamma^{-1} \approx 0.66\). From the slope of the smaller part of \(L/t^{1/3}\) of the single curve after the data collapse, the value of \(\alpha\) was estimated as \(\alpha \approx 0.50\). We find that the values of \(\alpha\), \(\beta\), and \(\gamma\) obtained by the data-collapsing method are consistent with the theoretical result obtained from the KPZ eq.\(^\text{10}\) and satisfy the FV scaling law given by eq. (5) because \(\gamma \approx 0\).

3.1.2 Eden model

The Eden model is used to reproduce the growth of cell colonies.\(^\text{10}\) The interfaces in the Eden model are known to have the scaling exponents \(\alpha \approx 0.5\) and \(\beta \approx 0.30\) in \((1+1)\) dimensions.\(^\text{9}\) There are several versions of the Eden model. Among them, we adopted the one which is referred to as version-C by Jullien and Boret.\(^\text{11}\) Initially at \(t = 0\), we place particles on all sites at \(k = 0\). The other sites are empty. Therefore, \(h(j,t = 0) = 0\) for all \(j\). The boundary condition for the height of the grid is periodic. The time evolution rule of version-C is summarized as follows: (i) An occupied site of the interfaces is chosen with equal probability. (ii) An empty site is selected equiprobably among the empty sites that are connected to the occupied site. (iii) The new particle is placed on the empty site.

Figure 4(a) shows a set of snapshots of \(h\) for several values of \(t\) for \(L = W = 512\). The time evolution of \(\sigma\) is plotted in Fig. 4(b) for several values of \(L = W\). The \(t-\sigma\) curve for each \(L = W\) was obtained by averaging over 500 runs. From this figure, the slope of the \(t-\sigma\) curve in the range of \(t \lesssim 4 \times 10^5\) indicates that \(\beta \approx 0.30\) when \(W = 1024\). Figure 5(a) shows \(L-\sigma\) plots for four different values of \(t\). Each curve in this figure was obtained by averaging over 100 runs. These \(L-\sigma\) curves can be collapsed onto a single curve, as shown in Fig. 5(b), when \(\beta \approx 0.30\) and \(\gamma^{-1} \approx 0.60\). The slope of the single curve after the data collapse is \(\alpha \approx 0.50\). These values
of \( \alpha \) and \( \beta \) are consistent with those in a previous paper,\(^7\) and it is easily confirmed that the values of \( \alpha, \beta, \) and \( \gamma \) satisfy the FV scaling law given by eq. (5) because \( \gamma \approx 0.\)

### 3.1.3 Directed percolation depinning model (DPD model)

The DPD model expresses the dynamic process of growing interfaces with quenched disorder such as water imbibition in paper. There are several versions of the DPD model. In our simulation, the DPD-2 version, introduced by Buldyrev \textit{et al.}\(^{3,12}\) was adopted. The model is explained as follows. We assign a random number \( \xi(j,k) \), which is distributed uniformly between 0 and 1, at each site. A fixed value \( p \) (0 \( \leq \) \( p \leq 1 \)) is introduced and is compared with \( \xi(j,k) \). If \( \xi(j,k) \) is smaller than \( p \), we place an obstacle on
this site so that no particle can be placed there during time evolution. It should be noted that a site containing an obstacle is not an empty site. Initially, a flat interface exists at $k = 0$ ($h = 0$). The boundary condition for the height of the grid is periodic. The iteration rule is summarized as follows. (i) We select a position on the baseline segment $c(j, 0)$, i.e., $j$, equiprobably. Among empty sites belonging to the selected position $j$, we select all sites that are connected to occupied sites. Then, we place a particle on each of the selected empty sites. (ii) We also place a particle on each of the empty sites below an occupied site. This rule is selected empty sites. (ii) We also place a particle on each of the selected positions $j$. We set $\beta \approx 0.66$ and $z^{-1} \approx 1.0$. The value (0.66) in both figures shows the slope of the guide line, which indicates the value of $\alpha$.

From a previous study, it is known that a moving interface is realized when $p \lesssim 0.47$. Hence, we set $p = 0.45$ in our simulation. Figure 6(a) shows a set of snapshots of $h$ as a function of $j$ for $L = W = 512$. The $t$-$\sigma$ plot obtained by averaging over 100 runs for several values of $L$ ($= W$) is shown in Fig. 6(b). It is found from this figure that the slope of the $t$-$\sigma$ curve in the range of $t = 10^4$ to $10^5$ gives $\beta \approx 0.66$ when $L = W = 1024$. Figure 7(a) shows $L$-$\sigma$ plots at four different values of $L$. Each curve in this figure was obtained by averaging over 100 runs. Using the data-collapsing method, these $L$-$\sigma$ curves can be collapsed onto a single curve, as shown in Fig. 7(b), when $\beta \approx 0.66$ and $z^{-1} \approx 1.0$. From the slope of the single curve after the data collapse, the value of $\alpha$ was estimated as $\alpha \approx 0.66$. These values of $\alpha$ and $\beta$ are consistent with those in a previous paper. It is found that the values of $\alpha$, $\beta$, and $z$ satisfy the FV scaling law [eq. (5)] because $\gamma \approx 0$.

### 3.2 Scaling relation at intermediate growth stage

In the original FV scaling, the growth exponent $\beta$ is defined only in the initial stage of surface growth (i.e., $t \ll \tau$). Thus, the original FV scaling cannot be applied to the growth state after the value of $\beta$ changes, such as the case of the Kardar–Parisi–Zhang equation with quenched noise (KPZQ eq.). On the other hand, in our scaling hypothesis, the scaling relations after the change in $\beta$ can be established.

The KPZQ equation

$$\frac{\partial h(x,t)}{\partial t} = v \frac{\partial^2 h(x,t)}{\partial x^2} + \lambda \left( \frac{\partial h(x,t)}{\partial x} \right)^2 + f + \eta_Q(x,h)$$

(13)

describes interface growth in a disordered medium. The meanings of $v$ and $\lambda$ are the same as those in the original KPZ eq. $f$ corresponds to a constant driving force. $\eta_Q(x,h)$ is a noise term that is independent of time, and it is called quenched noise. $\eta_Q(x,h)$ is assumed to have no correlation with respect to both $h$ and $x$. By discretizing eq. (13) for space and time, we obtain
for several values of $h$ (i.e., $h \sim h^0$) of KPZQ eq. for $L = 500$ and $W = 3000$. The time for each interface is shown on the right side of the figure. The plots are shifted vertically to avoid overlapping. (b) Time dependence of $\sigma$ for several values of $L (= W)$ in KPZQ eq. The values (0.60 and 0.33) show the slopes of the guide lines. Each result was obtained by averaging over 200 runs.

$$h(j, n + 1) = h(j, n) + \frac{\Delta t}{(\Delta x)^2} \left\{ v[h(j - 1, n) - 2h(j, n) + h(j + 1, n)] + \frac{\lambda}{8} [h(j + 1, n) - h(j - 1, n)]^2 + \Delta t [f + \eta_0(j, h)] \right\} + \Delta t \left[ f + \eta_0(j, h) \right].$$

The parameter values used in the simulation were $\Delta x = 1$, $\Delta t = 0.05$, $v = 1.0$, $\lambda = 4.0$, and $f = 0.05$. Regarding the noise term $\eta_0$ for each column $j$, a random number between 0 and 1 is assigned to each integer position of the height. For a noninteger position, $\eta_0$ is given by linear interpolation of the two nearest integer positions. The initial and boundary conditions are the same as those in the KPZ eq.

Figure 8(a) shows a set of snapshots of the fluctuation in $h$ for several values of $t$ for $L = 500$ and $W = 3000$. As a preliminary check, we plot $\sigma(L, t)$ as a function of $t$ for several values of $L (= W)$, as shown in Fig. 8(b). The $t-\sigma$ plot was obtained by averaging 200 runs. It is found that there exist two time regions, each having a different value of $\beta$: region I $\beta \approx 0.60 (10^0 \leq t \leq 10^2)$ and region II $\beta \approx 0.33 (t \geq 10^2)$. This result is consistent with those in previous reports. However, it seems that it has not yet been confirmed whether the FV scaling law is satisfied or not in region II. By the data-collapsing method, we estimated the values of the exponents for each scaling region. Figure 9(a) shows $L-\sigma$ curves for several $t$ in region I. Each curve in this figure was obtained by averaging over 100 runs. These curves can be collapsed onto a single curve, as shown in Fig. 9(b), when $\beta \approx 0.60$ and $z^{-1} \approx 0.80$. From the slope of the single curve after the data collapse, the value of $\alpha$ was estimated as $\alpha \approx 0.75$. These values of $\alpha$, $\beta$, and $z$ are consistent with the result of Csáthók and coworkers and satisfy the FV scaling law because $\gamma \approx 0$.

Similarly in region II, we plot $L-\sigma$ curves for several values of $t$, as shown in Fig. 10(a). Each curve in this figure was obtained by averaging over 100 runs. By the data-collapsing method, these curves are also collapsed onto a single curve, as shown in Fig. 10(b), when $\beta \approx 0.33$ and $z^{-1} \approx 0.44$, which are different from the values in region I. From the slope of the single curve after the data collapse, the value of $\alpha$ was estimated as $\alpha \approx 0.75$. It is found that the values of $\alpha$, $\beta$, and $z$ in region II also satisfy the FV scaling law given by eq. (5). Note that using our proposed extended scaling hypothesis, the scaling relations for different temporal scaling regimes can be established without any difficulty.

3.3 Cases when $\gamma \neq 0$

Our scaling relation is based on the asymptotic behavior
of $\sigma$ for $L$, and we do not consider its asymptotic behavior for $t$. Thus, by introducing a new exponent $\gamma$, our scaling relation is applicable to the case where $\sigma(L,t)$ continues to increase and is not saturated within a limited observation time. In this subsection, we report the results of a numerical model and a paper-wetting experiment where $\gamma \neq 0$.

### 3.3.1 Invasion percolation model (IP model)

Invasion percolation is a dynamic percolation process that represents the displacement of one fluid by another in a porous medium. There have been many applications of the invasion percolation process to various phenomena such as the displacement of wetting fluids by immiscible nonwetting fluids under gravity, and the aggregation of granular particles caused by the sweeping front motion. Here, we applied the invasion percolation process to interface growth by considering the following model. We assign a random number $\xi(j,k)$, which is distributed uniformly between 0 and 1, at each site. Initially the flat interface is located at $k = 0$ ($h = 0$). The boundary condition for the height of the grid is periodic. The iteration rule for placing a particle on a site is as follows. (i) We select the empty site that has the lowest random number $\xi$ among the empty sites connected to occupied sites. (ii) We place a particle on each of the empty sites below an occupied site. This rule is necessary to make the interface a single-valued function. We consider that this model corresponds to interface growth phenomena with a strong pinning effect. So far, the statistical behavior of interface growth obtained by this model appears to be poorly understood.

Figures 11(a) and 11(b) show a set of snapshots of $h$ as a function of $j$ for $L = W = 512$ and $t - \sigma$ plots for several values of $L$ ($= W$), respectively. Each $t - \sigma$ curve was obtained by averaging over 100 runs. It is found that the slope of the $t - \sigma$ curve in the range of $t = 5$–20 indicates that $\beta \approx 1.1$ when $W = 512$. Figure 12(a) shows $L - \sigma$ plots for four different values of $t$. Each curve in this figure was obtained by averaging over 200 runs. Using the data-collapsing method, these $L - \sigma$ curves can be collapsed onto a single curve, as shown in Fig. 12(b), when $\beta \approx 1.1$ and $z^{-1} \approx 1.05$. From the slope of the single curve after the data collapse, the value of $\alpha$ was estimated as $\alpha \approx 0.75$. It is found that the values of $\alpha$, $\beta$, and $z$ obtained by the data-collapsing method do not satisfy the FV scaling law given by eq. (5) because $\gamma = \beta - \alpha/z \approx 0.31 \neq 0$. Figure 13 shows the plot of $L$ vs $\sigma/t^{\gamma}$ for four different values of $t$. It is confirmed that the dependence of $\sigma$ on $t$ for small $L$ vanishes by rescaling $\sigma$ using the exponent $\gamma$. This implies that the nontrivial scaling described by the upper relation in eq. (9) holds in this case.
3.3.2 Paper-wetting experiment

Paper-wetting phenomena have been used to study the properties of interface growth in a disordered medium. There have been some reports on the scaling of surface growth in paper-wetting experiments.1,13 However, the confirmation of the FV scaling relation was insufficient in most cases, since the precise estimation of the value of $\beta$ was difficult from the slope of the $\ln\sigma(L, t)$ vs $\ln t$ plot. Here, we report the results of the scaling property obtained by our extended scaling hypothesis for a paper-wetting experiment. We used paper towels and water as the disordered medium and invading fluid, respectively. The size of each towel was $23.5 \times 27.5$ cm$^2$. We set a towel horizontally and soaked one of the short edges of the towel with water. A precise explanation of the setup of our experiment is given in a previous paper.25 Then, water invaded the towel, and interfaces between dry and wet domains grew. We took photographs of the whole towel during interface growth using a digital camera (CASIO, Tokyo). The size of images was $1200 \times 1600$ pixels (1 pixel $\approx 0.15$ mm). We examined the statistical properties of the growing interfaces obtained during paper wetting.

Figure 14(a) shows $L-\sigma$ curves for the paper-wetting experiment. Each curve in this figure was obtained by averaging over 10 trials. Using the data-collapsing method, these $L-\sigma$ curves can be collapsed onto a single curve, as shown in Fig. 14(b), when $\beta \approx 0.72$ and $z^{-1} \approx 0.73$. From the slope of the single curve after the data collapse, the value of $\alpha$ was estimated as $\alpha \approx 0.73$. It is noted that these values of $\alpha$ and $\beta$ are consistent with those of DPD model which is shown in §3.1.3 and refs. 1, 3, and 13. Furthermore, the value of $\alpha$ is also consistent with that of paper-wetting experiment reported previously.1,13 From this result, it is found that the values of $\alpha$, $\beta$, and $z$ do not satisfy the FV scaling law [eq. (5)] because $\gamma = \beta - \alpha/z \approx 0.19 \neq 0$. Figure 15 shows the $L$ vs $\sigma/\tau^\gamma$ plot for four different values of $t$. It is confirmed that the time dependence of $\sigma$ for small $L$ vanishes by rescaling $\sigma$ using the exponent $\gamma$. This again implies that the scaling relation, eq. (9), holds for small $L$.

4. Discussion and Conclusion

In this paper, a new scaling hypothesis for the roughness of growing interfaces corresponding to an extension of the Family–Vicsek (FV) scaling hypothesis has been proposed. Our scaling relation is imposed on the asymptotic behavior of $\sigma$ for $L$, and we do not consider its asymptotic behavior for $t$. Comparing eq. (3) with eq. (7), both equations are mathematically identical if eq. (5) is satisfied. Nevertheless, the important point is that the applicability of eq. (7) to results obtained from numerical simulation and experiments is significantly better than that of eq. (3). The advantages of our scaling hypothesis are summarized as follows. (A) $L$ is easily varied and the precise measurement of the $L$ dependence of $\sigma$ is possible. Thus, estimating the growth exponent $\beta$ is easier. (B) $\beta$ can be defined not only in the initial stage of interface growth but also in intermediate stages, where the value of $\beta$ may be different from that in the initial stage. (C) In some experimental cases, $\sigma(L, t)$ continues to increase and is not saturated within a limited observation time; FV scaling breaks down. In spite of that, a scaling argument is possible using our extended scaling hypothesis by introducing a new exponent $\gamma$. When $\gamma = 0$, the extended scaling is consistent with the FV scaling. Hence the exponent $\gamma$ can be considered as the deviation from the FV scaling.

Previously, we proposed the following asymptotic scaling behavior of $\sigma$ instead of the present relation given in eq. (9):

$$\sigma(L, t) = \begin{cases} \beta L^\beta, & t \ll \tau^*, \\ L^\gamma \tau^\gamma, & t \gg \tau^*, \end{cases}$$

$$\sigma(L, t) \sim \begin{cases} \beta L^\beta, & t \ll \tau^*, \\ L^\gamma \tau^\gamma, & t \gg \tau^*, \end{cases}$$

$$\sigma(L, t) \sim \begin{cases} \beta L^\beta, & t \ll \tau^*, \\ L^\gamma \tau^\gamma, & t \gg \tau^*, \end{cases}$$
where $\delta$ is a constant.\cite{25} In many numerical models and experiments including the ones presented here, the spatial interaction for $h(x)$ is considered to be local. Thus, the value of the exponent $\delta$ should be zero identically. Comparing eq. (15) with eq. (9), it is considered that our scaling relation presented in this paper is a modification of the previous scaling law holds. Furthermore (in §3.3), by introducing the exponent $\gamma$, the new scaling hypothesis enabled us to treat growing interfaces obtained by the IP model and the paper-wetting experiment, which do not satisfy the FV scaling law in eq. (5). We expect that our scaling hypothesis can be applied to a wide variety of interface growth phenomena in numerical models and experiments. In fact, our preliminary experiments strongly indicate that the bacterial colony formation also does not satisfy the FV scaling law. This will be published elsewhere in the near future.

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